Quantum Theory of Geometry III:
Non-commutativity of Riemannian Structures

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Abstract

The basic framework for a systematic construction of a quantum theory of Riemannian geometry was introduced recently. The quantum versions of Riemannian structures –such as triad and area operators– exhibit a non-commutativity. At first sight, this feature is surprising because it implies that the framework does not admit a triad representation. To better understand this property and to reconcile it with intuition, we analyze its origin in detail. In particular, a careful study of the underlying phase space is made and the feature is traced back to the classical theory; there is no anomaly associated with quantization. We also indicate why the uncertainties associated with this non-commutativity become negligible in the semi-classical regime.

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I. INTRODUCTION

This article is a continuation of [1,2] which will be referred to as papers I and II respectively. Unless otherwise specified, we will use the same notation and conventions.

Let us begin with a brief summary. In a non-perturbative treatment of quantum gravity, a preferred classical metric is not available. Therefore, one has to develop the appropriate field theory without any reference to a background geometry. One possibility is to use the canonical approach based on connections. Here, the configuration variable is an $SU(2)$ connection $A_a^i(\vec{x})$ on a three-manifold Σ (which serves as the kinematic arena). The momentum variable is a triad $E_i^a(\vec{x})$ with density weight one. (Indices $a, b, c, ...$ refer to the tangent space of Σ and $i, j, k, ...$ to the $su(2)$ Lie-algebra.) Since the classical configuration space $\mathcal{A}$ is the space of smooth connections on Σ, the quantum configuration space $\bar{\mathcal{A}}$ turns out to be a space of suitably generalized connections on Σ. To obtain the Hilbert space $\mathcal{H}$ of quantum states and geometric operators thereon, one needs a functional calculus on $\bar{\mathcal{A}}$ which also does not refer to a background metric (or any other field).

The necessary tools were developed in a series of papers by a number of authors [3–14]. (Much of the motivation for this work came from the 'loop representation' introduced earlier by Rovelli and Smolin [15].) It turns out that $\bar{\mathcal{A}}$ admits a natural diffeomorphism invariant measure $\mu_\circ$ [4–7,9] and the Hilbert space $\mathcal{H}$ can be taken to be the space $L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_\circ)$ of square-integrable functions on $\bar{\mathcal{A}}$. Physically, $\mathcal{H}$ represents the space of kinematic quantum states, i.e., the quantum analog of the full phase space. Using the well-developed differential geometry on $\bar{\mathcal{A}}$ [8], one can then define physically interesting operators on $\mathcal{H}$. In particular, one can introduce, in a systematic manner, operator-valued distributions $\hat{E}_i^a$ corresponding to the triads [1]. As in classical differential geometry, these are the basic objects of quantum Riemannian geometry. Specifically, operators corresponding to area, volume and length are constructed by regularizing the appropriate products of these triad operators [1,2,16]. (For related frameworks, see [17–20]).

Being density weighted, the triads E_i^a are duals of pseudo two-forms $e_{abi} := \eta_{abc} E_i^c$. In the quantum theory, therefore, one might expect that they should be smeared against Lie-algebra-valued test fields f_i with support on *two-dimensional surfaces*. This expectation turns out to be correct[1] [1]. However, somewhat surprisingly, the resulting operators $^2\hat{E}[S, f]$ turn out not to commute. For example, for operators smeared by two different test fields on the *same* two-surface, we have:

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[1]In this approach to quantum geometry, there is a remarkable synergy between geometry and analysis: in the regularization procedure, well-defined operators result when n-forms are integrated on n-manifolds. Thus, the operators that code information about connections are holonomies $\hat{h}[\alpha]$, obtained by integrating the connection one-forms along one dimensional curves. The triad two-forms are naturally regulated through a two dimensional smearing. This feature is deeply connected with the underlying diffeomorphism invariance of the theory. By contrast, in the quantum theory of Maxwell fields in Minkowski space-time, for example, using the geometrical structures made available by the background metric, one uses a three dimensional smearing for both connection one-forms and electric field two-forms.

$$[{}^2\hat{E}[S,f], {}^2\hat{E}[S,g] \neq 0 \tag{1.1}$$

if the Lie-bracket $[f,g]^i = \epsilon^i{}_{jk}f^j g^k$ fails to vanish. If the two operators are smeared along two distinct surfaces, the commutator is again non-zero if the two surfaces intersect and the Lie-bracket of the corresponding test fields is non-zero on the intersection.

This feature is at first surprising –even disturbing– because it implies that we can not simultaneously diagonalize all the triad operators. Could it be related to the fact that triads are only covariant rather than invariant under $SU(2)$ rotations? Would this non-commutativity disappear if one dealt only with manifestly gauge invariant objects? The answer is in the negative: the non-commutativity extends also to, e.g., the area operators which are gauge-invariant. In particular, if we have two surfaces S_1 and S_2 which intersect along a line, the corresponding area operators $\hat{A}[S_1]$ and $\hat{A}[S_2]$ fail to commute. The commutator $[\hat{A}[S_1], \hat{A}[S_2]]$ is non-zero only on those states which (in the terminology of paper I) have a four or higher valent vertex on the intersection. It is true that, heuristically, such states are 'non-generic'. Nonetheless, they constitute an infinite dimensional subspace of the space of gauge invariant states. Hence, even if we restrict ourselves to the gauge invariant context, the non-commutativity persists and makes it impossible to simultaneously diagonalize all the geometric operators. Thus, *the quantum Riemannian geometry that arises in this framework is genuinely non-commutative* and, at a fundamental level, one must face the Heisenberg uncertainties associated with geometric quantities. However, because the commutators fail to vanish only on certain rather special states, as one might expect, the quantum uncertainties turn out to be completely negligible in the semi-classical regime.

Nonetheless, the fact that the triad –and hence the metric– representation fails to exist in this approach is striking and was not fully appreciated in the early literature on the subject. It is important to understand its origin. Does this feature arise because of some subtleties associated with the classical Poisson brackets? Or, is this a quantum anomaly? In either case, what precisely are the underlying assumptions that lead to this non-existence of the triad representation? The purpose of this paper is to address these issues in a systematic fashion. We should emphasize that the consistency of the quantum theory is not in question here. The construction of the Hilbert space and the introduction of the operators has been carried out in a rigorous fashion. The regularization procedure is natural and tight. Rather, one wishes to reconcile the results of that analysis with one's intuition, particularly with what one knows about the phase space structure of the classical theory.

The paper is organized as follows. In section II, we will discuss certain subtleties associated with phase space structures: the 'obvious' choice of configuration and momentum variables turns out to be inappropriate in view of gauge and diffeomorphism invariance and, furthermore, the 'obvious' choice of Poisson brackets between the appropriate variables leads to inconsistencies. One must therefore find an appropriate substitute of the naive Poisson brackets. The required Lie algebra is presented in section III. We will find that the Lie-bracket structure of the classical theory simply mirrors that found in paper I for the quantum theory. Thus, there is *no* quantum anomaly. The main results are then examined from several angles to reconcile them with intuitive expectations. Section IV contains a summary and remarks. In particular, we elucidate why the non-commutativity of area operators does not lead to unwarranted uncertainties in the semi-classical regime.

II. PHASE SPACE STRUCTURES: SUBTLETIES

In section II A, we recall the structure of the classical phase space and fix notation. In section II B, we point out that the underlying diffeomorphism invariance now leads us to phase space variables which are rather different from those used, e.g., in the Maxwell theory in Minkowski space: now the configuration variables are Wilson loops and the momenta are triads smeared in *two* dimensions. From the viewpoint of standard phase space discussions, these functionals are 'singular' as they are obtained by smearing the basic canonical variables in one or two dimensions, rather than three. Consequently, one's naive expectation on the structure of their Poisson algebra may be incorrect. In II C, we will see that this is indeed the case: the naive Poisson brackets fail to satisfy the Jacobi identity.

A. Phase space

Fix an oriented, analytic[2] three-manifold Σ. We will assume that Σ is either compact or that the various fields satisfy suitable boundary conditions at infinity, the details of which will play no role in our analysis. Because Σ is three-dimensional and oriented, the principal $SU(2)$ bundle over Σ is trivial. Therefore, we can represent $SU(2)$ connections on the bundle by $su(2)$-valued one-forms $A_{aC}{}^D$, where a is the one-form index and C, D refer to the fundamental representation of $SU(2)$. For notational simplicity, we will often set $A_{aC}{}^D = A_a^i \tau_{iC}{}^D$, where the anti-Hermitian τ_i are related to the Pauli matrices σ_i via $2i\tau_i = \sigma_i$. (Thus, $-2\mathrm{Tr}\tau_i\tau^j = \delta_i^j$ and $[\tau_i, \tau_j] = \epsilon_{ijk}\tau^k$.) We will assume that all fields on Σ are smooth.

The configuration space $\mathcal{A}$ consists of all smooth connections A_a^i on Σ satisfying the boundary conditions. Thus, $\mathcal{A}$ is naturally an affine space. The phase space is the cotangent bundle over $\mathcal{A}$. The momenta are represented by smooth vector densities E_i^a of weight one on Σ, or equivalently, by the triplet of two-forms $e_{ab}^i = \eta_{abc}E^{ci}$. The fundamental Poisson brackets are given by:

$$\{A_a^i(\vec{x}), A_b^j(\vec{y})\} = 0 \quad ; \quad \{E_i^a(\vec{x}), E_j^b(\vec{y})\} = 0;$$
$$\{A_a^i(\vec{x}), E_j^b(\vec{y})\} = G\, \delta_a^b\, \delta_j^i\, \delta^3(\vec{x}, \vec{y}), \tag{2.1}$$

where $\vec{x}$ and $\vec{y}$ denote points on Σ and $G = 8\pi\, G_{\mathrm{N}}$, where G_{N} is Newton's constant. In particular, the triads Poisson-commute among themselves. This is why it is at first very surprising that the quantum triads fail to commute. The naive conclusion would be that the specific quantization is anomalous; vanishing Poisson brackets in (2.1) go over to non-trivial commutators. However, we will see in sections II B and II C that this is not the case.

For now, let us only note the meaning of the Poisson brackets. The fields A_a^i and E_i^a on Σ are *not* functions on the phase space. Therefore, the Poisson brackets (2.1) between

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[2]The assumption of analyticity is made only because it will simplify certain technicalities in sections II and III. We believe that the entire discussion can be carried over to the smooth and piecewise linear categories by appropriate modifications along the lines of [9,21] respectively.

them can only be interpreted in the distributional sense. That is, given any vector density v_i^a which takes values in the dual of the $su(2)$ Lie algebra, and a one-form f_a^i which takes values in the $su(2)$ Lie algebra, we can naturally define smooth functions

$$^3A[v] := \int_\Sigma \mathrm{d}^3x \, A_a^i(\vec{x})v_i^a(\vec{x}), \qquad ^3E[f] := \int_\Sigma \mathrm{d}^3x \, E_i^a(\vec{x})f_a^i(\vec{x}) \qquad (2.2)$$

on the phase space, where the superscript 3 makes it explicit that we have used three dimensional test fields to smear the basic variables. Equation (2.1) is then just a short-form for the following Poisson brackets between these well-defined configuration and momentum *functions*:

$$\{^3A[v], {}^3A[v']\} = 0 \quad ; \quad \{^3E[f], {}^3E[f']\} = 0;$$
$$\{^3A[v], {}^3E[f]\} = G \int_\Sigma \mathrm{d}^3x \, v_i^a f_a^i. \qquad (2.3)$$

A priori, relations (2.1) do not say anything about the Poisson brackets between 'singular' functions obtained by integrating A_a^i and E_i^a by distributional smearing fields.

In linear field theories in Minkowski space, there is no need to consider 'singular' smearings. Indeed, one generally begins with the Abelian algebra generated by finite complex linear combinations of finite products of configuration variables $^3A[v]$. These are generally referred to as cylindrical functions on the configuration space $\mathcal{A}$. Because of the underlying linear structure of $\mathcal{A}$, one can apply the standard Kolmogorov [22] theory to integrate these functions. Thus, to construct the quantum theory, one has to select a suitable cylindrical measure, define an Hermitian inner product between cylindrical functions and Cauchy complete the space to obtain the quantum Hilbert space $\mathcal{H}$. On this Hilbert space, the configuration operators $^3\hat{A}[v]$ act naturally by multiplication. Finally, to complete the kinematic set-up, one defines the action of the momentum operators $^3\hat{E}[f]$ so that the Poisson brackets (2.3) are taken over to $i\hbar$ times the commutators. (Typically, this action is obtained by studying the action of the Hamiltonian vector fields generated by $^3E[f]$ on cylindrical functions.)

In the present case, however, the kinematic symmetries of the theory –the $SU(2)$ gauge invariance and the Σ-diffeomorphism invariance– make this strategy unsuitable. (See, e.g., [23] for details.) More precisely, the simple configuration variables in (2.2) fail to be gauge covariant and are therefore unsuitable in the non-Abelian context (unless one manages to fix the gauge completely). Furthermore, there is also a problem with respect to the diffeomorphism invariance: the above cylindrical functions fail to be integrable with respect to any of the known diffeomorphism invariant (generalized) measures. Hence, the simple Poisson algebra (2.3) is no longer suitable as the starting point for quantization. One has to find an appropriate substitute.

B. Appropriate phase space variables

The strategy [24,15,3] that has been most successful is to construct the configuration observables through holonomies of connections. Perhaps the simplest possibility is to use, in

place of $^3A[v]$, the Wilson loop functionals, i.e. traces of holonomies of connections around closed, piecewise analytic loops, $T[\alpha]$:

$$T[\alpha] := \frac{1}{2}\operatorname{Tr}\mathcal{P}\exp\left(-\oint_\alpha A_a \mathrm{d}s^a\right) \tag{2.4}$$

where $\mathcal{P}$ stands for 'path ordered'. The algebra generated by these functions is called the holonomy algebra [3].

While this construction of the algebra is simple and direct, in order to do functional analysis on this space of functions of connections, one has to introduce, in addition, considerable technical machinery. This involves the introduction of the appropriate notion of 'independent loops' and techniques for decomposing arbitrary piecewise analytic loops into a finite number of independent ones. (For details, see [4]). Furthermore, by construction, all elements of the holonomy algebra are gauge invariant. Since we wish to examine the action of canonical transformations generated by triad functions –which are only gauge covariant– we need a larger arena to work with. We will therefore proceeds as follows [5,8].

Denote by γ a graph in Σ with N analytic edges. (For our purposes, an intuitive understanding of these notions will suffice. For precise definitions, see, e.g., [1,8].) Let us fix a global cross-section of our principal $SU(2)$ bundle. Then, every connection A in $\mathcal{A}$ associates with each edge e of γ an element $h_e(A)$ of $SU(2)$, its holonomy along the edge e. Therefore, given any complex-valued, smooth function c, on $[SU(2)]^N$,

$$c: [SU(2)]^N \longrightarrow \mathbf{C}$$

we acquire a function C_γ on $\mathcal{A}$:

$$C_\gamma(A) := c(g_1, \dots g_N) \tag{2.5}$$

(Since the function on $\mathcal{A}$ depends not only on γ but also on our choice of c, strictly, it should be denoted as $C_{\gamma,c}$. However, For notational simplicity, we will only retain the suffix γ.) These configuration variables capture only 'finite dimensional pieces' from the infinite dimensional information in the connection field A_a^i; they are sensitive only to what the connection does on the edges of the graph γ. Following the terminology used in linear field theories, they are called *cylindrical functions*. The space of cylindrical functions associated with any graph γ is denoted by Cyl_γ. The Wilson loops functionals T_α associated with closed loops α that lie entirely in γ clearly belong to Cyl_γ. (By suitably restricting the form of functions c, we can make Cyl_γ the algebra generated by these Wilson loop functionals. However, we shall not require this.) The space Cyl_γ is quite 'small': it only contains the configuration observables associated with the 'lattice gauge theory' defined by the graph γ. However, as we vary γ, allowing it to be an arbitrary graph (with a finite number of analytic edges), we obtain more and more configuration variables. Denote the union of all these by Cyl. This is a very large space. In particular, it suffices to separate points of $\mathcal{A}$. *We will take* Cyl *to be the space of our configuration variables.*

Next, we wish to consider the triads. Let us begin with $^3E[f]$. What are the Poisson brackets between $^3E[f]$ and functions in Cyl? Since the Hamiltonian vector field of $^3E[f]$ is well-defined, the calculation is easy to perform. One obtains:

$$\{C_\gamma,\ ^3E[f]\} = G\int_\Sigma \mathrm{d}^3x\, f_a^i(\vec{x})\frac{\delta C_\gamma}{\delta A_a^i(\vec{x})}$$

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$$= G \sum_{I=1}^{N} \int_{e_I} \mathrm{d}t_I \, \dot{e}_I^a(t_I) f_a^i(e_I(t_I)) \, [h(1,t_I)\tau_i h(t_I,0)]_B^A \, \frac{\partial c}{\partial h_{IB}^A} \, , \qquad (2.6)$$

where t_I is a parameter along the edge e_I that runs between 0 and 1, and $h(t_I,0)$ is the holonomy along e_I from $t_I = 0$ to $t_I = 1$. Thus, as expected, the Poisson bracket is well-defined. However, unfortunately, the result is *not* a cylindrical function: because of the t_I integration involved, the right side requires the knowledge not just of a finite number of holonomies, $h_1, \dots, h_N$, but of N-continuous parameter worth of them, $h_I(t)$.

Recall from section II A that a standard quantization strategy is to construct the Hilbert space of states from configuration variables. In the resulting quantum theory, the action of the momentum operators is then dictated by the action, on configuration variables, of the Hamiltonian vector fields generated by momenta. Therefore, it is highly desirable that the configuration variables be closed under this action. Unfortunately, this is not possible with $^3E[f]$ as our momenta.

Note, however, that a drastic simplification occurs if the smearing field f_a^i is distributional with two-dimensional support. Let us, for definiteness introduce local coordinates on Σ and assume that the smearing field has the form:

$$^\epsilon f_a^i(\vec{x}) = h_\epsilon(z)(\nabla_a z)f^i(x,y),$$

where $h_\epsilon(z)$ tends to $\delta(z)$ in the limit that the parameter ϵ goes to zero. Then, in the limit as ϵ goes to zero, the continuous integral over t_I in (2.6) collapses to a finite sum over the points p at which e_I intersects the surface $z = 0$ and the right side is again a cylindrical function.

Let us evaluate this limit explicitly. At each of the intersection points p, one has a number of edges, e_{I_p}, with $I_p = 1, 2, \dots, n_p$. If all these edges are oriented away from the surface, we have:

$$\lim_{\epsilon \mapsto 0} \{C_\gamma, \, ^3E[^\epsilon f]\} = \frac{G}{2} \sum_p \sum_{I_p} \kappa(I_p) \, f^i(p)[h_{I_p}\tau_i]_B^A \, \frac{\partial c}{\partial h_{I_p B}^A} \, , \qquad (2.7)$$

where the constant $k(I_p)$ equals $+1$ if the edge lies entirely above S, -1 if it lies below S and 0 if it is tangential to S. If an edge is oriented towards the surface, the only change is that the factor $[h_{I_p}\tau_i]$ is replaced by $[\tau_i h_{I_p}]$. Finally, note that

$$[h_{I_p}\tau_i]_B^A \, \frac{\partial c}{\partial h_{I_p B}^A}$$

is just the result of the action of the *ith* left invariant vector field on the I_pth argument of the function $c(h_1, \dots, h_{I_p}, \dots)$ on $[SU(2)]^N$. Similarly, if an edge I_p is oriented towards the surface, we get the action of the *ith* right invariant vector field. The details of this argument can be found in section 3.1 of [1]. Here, we only note that there are no contributions from edges which lie in the limiting surface $z = 0$. For, if an edge e_I is tangential, we have $\dot{e}_I^a \nabla_a z = 0$ before taking the limit $\epsilon \mapsto 0$. Hence these edges make no contribution; $\kappa(I_p)$ vanishes in this case.

Thus, if we use distributional test-fields with two dimensional support and evaluate the Poisson bracket of the resulting smeared triad with cylindrical functions via a limiting

procedure, we find that the Poisson bracket closes on Cyl. Hence, it is natural to use two-dimensionally smeared triads

$$^2E[S, f] := \int_S e_{abi} f^i \mathrm{d}S^{ab} \tag{2.8}$$

as the momentum variables. However, the limiting procedure used to pass from three dimensionally smeared triads to the two dimensionally smeared ones is technically subtle and the result is not well-defined unless appropriate regularity conditions are imposed.

Let us spell these out. The test fields f^i must, as usual, be at least continuous. Conditions on the surface S are less transparent. First, the action (2.7) may not be well-defined if the surface S and the graph γ have an infinite number of intersections. This is easily avoided by demanding that S should also be analytic. For, each of our graphs γ has only a finite number of analytic edges and an analytic curve either intersects an analytic surface only at a finite number of points or it is tangential to it. Thus, analyticity of S will ensure that the sum on the right side of (2.9) has only a finite number of terms. Even then, however, ambiguities arise if the surface S has a boundary and an edge of the graph intersects S at one of the points on the boundary. Therefore, we are led to require that S does not have a boundary. (These regularity conditions were also necessary in [1] to ensure that the two-dimensionally smeared triad *operator* be well-defined in the quantum theory.) The last condition arises because the fields (A_a^i, E_i^a) spanning the classical phase space are all smooth. Because of this, if a surface S_2 is obtained from S_1 simply by deleting a set of measure zero, they define the same functions (2.8) on the phase space. However, since the right side of (2.7) involves distributions, the value one obtains for S_1 may be different from that for S_2; some edges of γ may intersect S_1 precisely at those points which are missing in S_2. Therefore, the limiting procedure can lead to well defined Poisson brackets only if we remove this ambiguity. We will do so by restricting the permissible surfaces: we will only consider surfaces S of the type $S = \bar{S} - \partial\bar{S}$, where $\bar{S}$ is any compact, analytic, 2-dimensional sub-manifold of Σ possibly with boundary. This condition ensures that S has no "missing points in its interior".

With these technicalities out of the way, let us now associate with each analytic surface S in Σ of the above type and each test field f^i thereon a momentum variable $^2E[S, f]$ on the phase space via (2.8). Following (2.7), let us set the Poisson bracket between these momenta and the configuration functions in Cyl to be:

$$\{C_\gamma, {}^2E[S, f]\} = \frac{G}{2} \sum_p \sum_{I_p} \kappa(I_p) f^i(p) X_{I_p}^i \cdot c \tag{2.9}$$

where $X_{I_P}^i \cdot c$ is the result of the action of the i-th left (resp. right) invariant vector field on the I_p-th copy of the group if the I_p-th edge is pointing away from (resp. towards) the surface S. Note the structure of the right hand side. The result is non-zero only if the graph γ used in the definition of the configuration variable C_γ intersects the surface S used to smear the triad. If the two intersect, the contributions arise from the action of right/left invariant vector fields on the arguments of c associated with the edges at the intersection.

It only remains to specify the Poisson brackets between the new momentum variables, $^2E[S, f]$. In view of (2.1), the obvious choice, assumed implicitly in some of the early literature, is to set:

$$\{^2E[S, f], \ {}^2E[S', f']\} = 0. \tag{2.10}$$

With this assumption, the Poisson brackets between all our fundamental configuration and momentum variables are specified. If these brackets satisfy the Jacobi identity, the resulting Lie algebra would offer a point of departure for the quantum theory.

C. Problem with the naive Poisson algebra

However, as we will see in this sub-section, the brackets (2.9) and (2.10) are in fact inconsistent with the Jacobi identity. Hence, one must develop a new quantization strategy. This task will be carried out in section III.

For simplicity, we will illustrate the problem with specific Poisson brackets. Fix a two-surface S, two smearing fields, f^i and g^i thereon, and a closed loop α which has a finite segment tangential to S as in figure 1. The Poisson brackets of interest to us are those between the momentum variables ${}^2E[S,f]$ and ${}^2E[S,g]$ and the configuration variable $C_\alpha :=$ T_α. (Here, $T_\alpha(A) = \frac{1}{2}\mathrm{Tr}\, h_\alpha$, where the trace of holonomy is taken in the fundamental representation.) For concreteness, we parametrize and orient the loop α as in the figure. We will present the calculation in some detail because the final result is quite surprising.
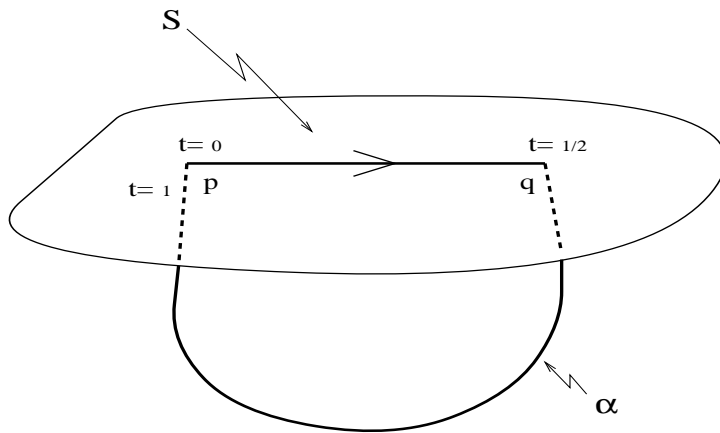
Fig. 1 The loop α has two analytic segments, one which lies in the surface S and the other which lies entirely below S. For definiteness, the loop α has been so parameterized that the parameter t runs from 0 to $\frac{1}{2}$ along the segment in S and from $\frac{1}{2}$ to 1 for the other segment. The surface is oriented so that the segment $t \in [1/2, 1]$ lies 'below' S.

Let us begin by computing the Poisson bracket $\{\mathrm{T}_\alpha, {}^2E[S,f]\}$ which is used repeatedly in the rest of the calculation. We have:

$$
\{\mathrm{T}_\alpha, {}^2E[S,f]\} = G \int \mathrm{d}^3x\, f_a^i(\vec{x})\, \frac{\delta \mathrm{T}_\alpha}{\delta A_a^i(\vec{x})}
$$

$$
= G \int_S \mathrm{d}S^{ab} \eta_{abc}\, f^i(\vec{s}) \int_0^1 \mathrm{d}t\, \dot{\alpha}^c(t)\, \delta^3(\vec{s}, \vec{\alpha}(t))\, \mathrm{Tr}[h_\alpha(1,t)\tau^i h_\alpha(t,0)], \qquad (2.11)
$$

where $\vec{s}$ denotes a generic point in S.

Therefore, for T_α and S as in figure 1, we have

$$\{T_\alpha,\, {}^2E[S,f]\} = G \int_S \mathrm{d}S^{ab}\, \eta_{abc} f^i \left[\int_0^{1/2} \mathrm{d}t\, \dot\alpha^c(t)\delta^3(\vec{s},\vec\alpha(t))\, \mathrm{Tr}[h_\alpha(1,t)\tau^i h_\alpha(t,0)] \right.$$

$$\left. + \int_{1/2}^1 \mathrm{d}t\, \dot\alpha^c(s)\delta^3(\vec{s},\vec\alpha(t))\mathrm{Tr}[h_\alpha(1,t)\tau^i h_\alpha(t,0)] \right]$$

$$= -\frac{G}{2} f^i(q)\mathrm{Tr}[h_\alpha(1/2,1/2)\tau^i] + \frac{G}{2} f^i(p)\mathrm{Tr}[\tau^i h_\alpha(1,0)] \qquad (2.12)$$

where we have used the standard convention, $\int_0^\infty \mathrm{d}x\, \delta(x) = \frac{1}{2}$, which can be justified from general consistency considerations (see, e.g. [1]). Therefore we have,

$$\{\{T_\alpha,\, {}^2E[S,f]\},\, {}^2E[S,g]\} = \frac{G^2}{4} f^i(q)g^j(q)\, \mathrm{Tr}[h_\alpha(1/2,1/2)\tau^i\tau^j]$$

$$- \frac{G^2}{4} f^i(q)g^j(p)\, \mathrm{Tr}[\tau^j h_\alpha(1,1/2)\tau^i h_\alpha(1/2,0)]$$

$$+ \frac{G^2}{4} f^i(p)g^j(p)\, \mathrm{Tr}[\tau^j \tau^i h_\alpha(1,0)]$$

$$- \frac{G^2}{4} f^i(p)g^j(q)\mathrm{Tr}[\tau^i h_\alpha(1,1/2)\tau^j h_\alpha(1/2,0)]. \qquad (2.13)$$

We are now ready to evaluate the left hand side of the Jacobi identity. We have:

$$J := \{\{{}^2E[S,f],\, {}^2E[S,g]\},\, T_\alpha\} + \{\{T_\alpha,\, {}^2E[S,f]\},\, {}^2E[S,g]\} + \{\{{}^2E[S,g],\, T_\alpha\},\, {}^2E[S,f]\}$$

$$= \{\{{}^2E[S,f],\, {}^2E[S,g]\},\, T_\alpha\} + \frac{G^2}{4} f^i(q)g^j(q)\, \mathrm{Tr}[h_\alpha(1/2,1/2)(\tau^i\tau^j - \tau^j\tau^i)]$$

$$+ \frac{G^2}{4} f^i(p)g^j(p)\, \mathrm{Tr}[(\tau^j\tau^i - \tau^i\tau^j)h(1,0)]$$

$$= \{\{{}^2E[S,f],\, {}^2E[S,g]\},\, T_\alpha\} + \frac{G^2}{4} f^i(q)g^j(q)\epsilon^{ijk}\, \mathrm{Tr}[h_\alpha(1/2,1/2)\tau^k]$$

$$+ \frac{G^2}{4} f^i(p)g^j(p)\epsilon^{ijk}\mathrm{Tr}[\tau^k h_\alpha(1,0)] \qquad (2.14)$$

By inspection, the sum of the last two terms is generically non-zero. Hence, the Jacobi identity will be violated if we *demand* that the Poisson bracket $\{{}^2E[S,f],\, {}^2E[S,g]\}$ between momenta must vanish. Thus, the bracket defined by (2.9) and (2.10) fails to be a Lie bracket and can not serve as the starting point for quantization.

Let us summarize. The kinematical symmetries of the theory suggest that we use as configuration variables elements of Cyl, based on holonomies along curves, rather than the three-dimensionally smeared functions ${}^3A[v]$ of section II A. This is a viable strategy because a (background-independent) functional calculus is readily available on Cyl. It is then natural to use two-dimensionally smeared triads, ${}^2E[S,f]$ as momenta because their Hamiltonian vector fields map Cyl to itself. Our task then is to define a consistent kinematical framework using these variables. Since the action of the Hamiltonian vector fields generated by ${}^2E[S,f]$ is well-defined on Cyl, the Lie-bracket between our configuration and momentum variables is unambiguous; it is given by (2.9). However, if we now require –as seems natural at first– that

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the momentum variables should Poisson-commute, the Jacobi identity is violated. Thus, we do not have a Lie algebra which can serve as the point of departure for quantum theory. Hence a new strategy is needed.

Remark: In the early literature on the subject, manifest $SU(2)$-gauge invariance was often at the forefront. The momentum functions were then taken to be traces of products of e^i_{ab} and holonomies (the T^0_α and the T^a_α variables of Rovelli and Smolin [15]) or smeared versions thereof (the so-called 'strip variables', associated with foliated, 2-dimensional strips (see, e.g., [14])). They are again linear in the triads e^i_{ab} but also depend on the connections A^i_a to ensure gauge invariance. A careful examination shows that the analog of problem with the Jacobi identity we just discussed exists also in that setting. Thus, the problem is *not* an artifact of our use of non-gauge invariant variables.

III. SOLUTION: A CONSISTENT LIE ALGEBRA

An algebraic approach a la Dirac is best suited to quantization especially because we wish to use a restricted class, Cyl, of smooth functions on the classical phase space as our configuration variables. We can begin by associating configuration operators with elements of Cyl and momentum operators with $^2E[S, f]$. To construct the quantum algebra, however, we need to specify the commutators between these operators. In the final theory, the configuration operators always act by multiplication and must therefore commute among themselves. The commutator of the momentum and configuration operators are also unambiguous; they are dictated by the bracket (2.9). Thus, what is needed is the commutator of the momentum operators among themselves.

In section III A, we will see that one can extract this information by exploiting the fact that our phase space has a cotangent bundle structure. That is, one can define brackets between the classical momentum variables $^2E[S, f]$ such that a *true* Lie-algebra results. However, it turns out that not all elements of this Lie-algebra can be represented as *functions* on the phase space. Nonetheless, we will see that the Lie algebra does serve as a viable point of departure for quantization. Furthermore, the resulting quantum algebra is precisely the one used in Refs [1-14]. Thus there is no anomaly in quantization; the *classical* brackets between momentum variables already fail to vanish. The origin of this non-commutativity is discussed in section III B.

A. Quantization Strategy: Exploiting the cotangent bundle structure

As we saw in section II A, our phase space is a cotangent bundle over the configuration space $\mathcal{A}$ of connections. It is therefore natural to try to repeat the strategy one uses for quantization of simple systems where the classical phase space Γ is a cotangent bundle over a finite dimensional manifold, say $\mathcal{C}$.

Let us first review that situation briefly (see, e.g., [25,26]). For such systems, one begins with the space $\mathcal{F}$ of suitably regular functions f on $\mathcal{C}$. Elements of $\mathcal{F}$ can be lifted to Γ to yield phase space functions which are independent of momenta. These are the configuration variables. The momentum variables $M(q, p)$ are functions on Γ which are linear in momenta (i.e. in the fibers of Γ): $M_V(q, p) = V^a p_a$ for *some* vector field V^a on $\mathcal{C}$. Thus, there

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is a natural isomorphism between the space of momentum variables and the space $\mathcal{V}$ of suitably regular vector fields on $\mathcal{C}$. The Poisson brackets between these elementary phase space functions (which yield the commutators between the elementary quantum operators) are given by:

$$\{f, f'\} = 0 \; ; \quad \{f, M_V\} = V \cdot f$$
$$\{M_V \, M_{V'}\} = M_{[V',V]} \,, \tag{3.1}$$

where $V \cdot f$ is the action of the vector field V on the function f and $[V, V']$ is the commutator of the two vector fields. Note that these operations refer only to the structure of the configuration space $\mathcal{C}$ rather than the phase space Γ. They mirror the natural Lie algebra structure that exists on pairs (f, V) of functions and vector fields on the configuration space $\mathcal{C}$:

$$[(f, V), (f', V')] = (V' \cdot f - V \cdot f', [V', V]) \tag{3.2}$$

Thus, in effect, in the quantum theory one associates configuration operators with elements of $\mathcal{F}$ and momentum operators with elements of $\mathcal{V}$ and the commutators between these operators are dictated by the natural Lie bracket (3.2) which refers only to $\mathcal{C}$. This description is completely equivalent to the one in terms of Poisson brackets (3.1) but does not directly refer to operations on the phase space Γ.

Let us now return to the problem at hand. Now the space $\mathcal{A}$ of (suitably regular) connections plays the role of $\mathcal{C}$ and the space Cyl of cylindrical functions plays the role of $\mathcal{F}$. Thus, to complete the Lie algebra, we need to isolate the analog of $\mathcal{V}$, the space suitable vector fields on $\mathcal{A}$. As the above discussion suggests, this task can be completed by examining the momentum variables $^2E[S, f]$. Indeed, it follows from (2.9) that, given the ring Cyl of cylindrical functions, we can naturally associate a vector field $X_{S,f}$ with the momentum variable $^2E[S, f]$:

$$^2E[S, f] \mapsto X_{S,f} \quad ; \quad X_{S,f} \cdot C_\gamma = \frac{1}{2} \sum_p \sum_{I_p} \kappa(I_p) \, f^i(p) \, X^i_{I_p} \cdot c \tag{3.3}$$

for any cylindrical function C_γ based on a graph γ. However, since $\mathcal{A}$ is infinite dimensional, it is important to specify the sense in which $X_{S,f}$ is a vector field: $X_{S,f}$ *is a derivation on the ring of cylindrical functions.* That is,

$$X_{S,f} : \mathrm{Cyl} \longrightarrow \mathrm{Cyl}$$

such that the map is linear and satisfies the Leibnitz rule:

$$X_{S,f} \cdot (C_\gamma + \lambda C'_\gamma) = X_{S,f} \cdot C_\gamma + \lambda X_{S,f} \cdot C'_\gamma$$
$$X_{S,f} \cdot (C_\gamma C'_\gamma) = C_\gamma \, X_{S,f} \cdot C'_\gamma + (X_{S,f} \cdot C_\gamma) \, C'_\gamma \tag{3.4}$$

for all cylindrical functions[3] C_γ and C'_γ and complex numbers λ. Finally, note that the

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[3]Note that there is no loss of generality in assuming that two cylindrical functions are cylindrical with respect to the same graph. Given C_γ and $C'_{\gamma'}$ where γ and γ' are distinct graphs, one can just consider a larger graph γ'' which contains all the vertices and edges of the two graphs. Then, the two given functions are cylindrical with respect to γ''.

commutator between two derivations on Cyl is well-defined and is again a derivation on Cyl. Furthermore, these derivations form a Lie-algebra; the commutator bracket automatically satisfies the Jacobi identity.

Hence, it is natural to use for $\mathcal{V}$ the vector space of derivations on Cyl generated by $X_{S,f}$. That is, $\mathcal{V}$ will be the vector space consisting derivations on Cyl resulting from finite linear combinations and a finite number of commutators of $X_{S,f}$. With this choice of $\mathcal{V}$ and with $\mathcal{F} = \mathrm{Cyl}$, the analog

$$\Big[(C_\gamma, X_{S,f}), (C'_{\gamma'}, X_{S',f'})\Big] = \Big(X_{S',f'} \cdot C_\gamma - X_{S,f} \cdot C'_{\gamma'}, \; [X_{S',f'}, X_{S,f}]\Big) \tag{3.5}$$

of (3.2) is a Lie-bracket for all cylindrical functions C_γ and $C_{\gamma'}$ and vector fields $X_{S,f}$ and $X_{S',f'}$ in $\mathcal{V}$. This is the Lie-algebra we were seeking. To go over to the quantum theory, with each element of Cyl, we can associate a configuration operator and with each element of $\mathcal{V}$, a momentum operator. The commutators between these operators can be taken to be $i\hbar$ times the classical Lie bracket (3.5). Furthermore, it is transparent from paper I that this quantum algebra is faithfully represented by operators on $\mathcal{H} = L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_\circ)$. Thus, the quantum theory of refs [1-14] in fact results from 'quantization of this classical Lie-algebra'. In particular, there is no anomaly associated with this quantization.

Let us summarize. For simple finite dimensional systems, there are two equivalent routes to quantization, one starting from the Poisson algebra of configuration and momentum functions on the phase space and the other from functions and vector fields on the configuration space. It is the second that carries over directly to the present approach to quantum gravity.

However, there are some important differences between the situation in the present case and that in simple finite dimensional examples. We will conclude this sub-section with two remarks on these differences.

i) In finite dimensional examples, $\mathcal{V}$ is generally taken to be the space of *all* smooth vector fields on the configuration space. Here, on the other hand, we only considered those derivations which can be generated from the basic vector fields $X_{S,f}$ by taking their finite linear combinations and a finite number of Lie brackets. What motivated this restriction? Could we have allowed *all* derivations on Cyl and still obtained a Lie algebra? The answer is in the affirmative. However, that procedure would have been inconvenient for two reasons.

First, whereas the $X_{S,f}$ are in one to one correspondence with the momentum functions $^2E[S, f]$ of (2.8) on the phase space, as we will see below, a generic derivation need not correspond to *any* phase space function. In the quantization procedure, on the other hand, it is convenient –and, for analyzing the classical limit, even essential– to have a correspondence between 'elementary operators' with which one begins and phase space functions [25,26]. Second, if one allows *all* derivations, one must specify relations between them which are to carry over to the quantum theory. Indeed, even in finite dimensional systems, such relations exist and give rise to certain anti-commutation relations which ensure that the operators corresponding to functions f and vector fields V and fV are correctly related [25,26]. In the present case, the task of specifying all such relations would have been formidable. By starting with a 'small' class of vector fields $X_{S,f}$, we avoid both sets of difficulties in one stroke. (The strategy we chose is, in some ways, analogous to the text-book treatment of quantization of a particle in the Euclidean space, where one builds the Lie-algebra from just constant vector fields $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$.)

ii) In the finite dimensional case, there is a one to one correspondence between suitably regular vector field V^a and momentum functions $V^a p_a$ on the phase space. In the present case, this correspondence continues to hold for the basic vector fields $X_{S,f}$ which generate $\mathcal{V}$. However, it does not extend to general elements of $\mathcal{V}$. A simple example is provided by the commutator $[X_{S,f}, X_{S',f'}]$ where S and S' intersect on a one dimensional line. It follows immediately from (3.3) that the commutator is again a derivation on Cyl but its action is non-trivial *only* on graphs with edges passing through the intersection of S and S'. That is, while the vector fields $X_{S,f}$ and $X_{S',f'}$ each have a two-dimensional support, the commutator has only one dimensional support. In particular, therefore, it can not be expressed as finite a linear combination of our basic vector fields and does not define a linear combination of momentum functions $^2E[S,f]$.

One might imagine going around this difficulty by extending the definition of momentum functions. For instance, one might associate functions $^2E[S,f]$ and vector fields $X_{S,f}$ not only with analytic surfaces S but also piecewise analytic ones. This strategy brings with it additional complications because the $\kappa(I_p)$ in (2.7) are now ambiguous for edges e_I passing through the "corners" at which the piecewise analytic S fails to be analytic. In simple situations, one can give a recipe to remove these ambiguities. Then, the commutator can be expressed as a linear combination of vector fields $X_{S^{(J)},f^{(J)}}$ associated with piecewise analytic surfaces $S^{(J)}$ constructed from S and S'. However, because the intersection of S and S' is one dimensional and fields E_i^a constituting the phase space are all smooth, the corresponding linear combination of momentum functions $^2E[S^{(J)}, f^{(J)}]$ simply vanishes! Thus, even after extending the definitions of momentum functions and basic vector fields, one can not establish a one to one correspondence between the vector fields and momentum functions. Therefore, we have refrained from carrying out this extension.

Note that, in spite of these differences, the quantization strategy is sound. The classical Lie algebra –which leads to the algebra of quantum operators– is generated by functions C_γ on $\mathcal{A}$ and vector fields $X_{S,f}$ on $\mathcal{A}$. Each of these generators defines, in a one to one fashion, functions $f(A,E) := C_\gamma(A)$ and $M(A,E) := \int_S e_{abi} f^i \mathrm{d}S^{ab}$ on the phase space. These functions are complete, i.e., suffice to separate the points of the phase space. To promote a classical observable to a quantum operator, we can first express it in terms of these basic functions and then carry the expression over to the quantum theory. The procedure carries with it only the standard factor ordering and regularization ambiguities. Thus, because the set of generators of the classical Lie-algebra is 'sufficiently large', the fact that some derivations do not correspond to phase space functions does not create an obstacle.

B. Origin of non-commutativity

Since we now have the Lie algebra on which quantization can be based, we can probe the origin of non-commutativity of the two-dimensionally smeared triad operators defined in [1]. At the classical level, the key question is of course whether the vector fields $X_{S,f}$ commute on Cyl. For definiteness, let us choose the orientation of the edges of γ in such a way that all the edges intersecting S are outgoing at the intersection points. Then the action of $X_{S,f}$ on any cylindrical function C_γ based on γ is given by

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$$X_{S,f} \cdot C_\gamma = \frac{1}{2} \sum_p f_i(p) \left(\sum_{I_p^u} X_{I_p^u}^i - \sum_{I_p^d} X_{I_p^d}^i \right) \cdot c. \tag{3.6}$$

Here the superscript u ('up') refers to edges which lie 'above' S and d ('down') to the edges which lie below (since Σ and S are both oriented, this division of edges can be made unambiguously) and, since the edges are all 'outgoing', X^i are the left-invariant vector fields (on the corresponding copy of the group in the argument of c.) It is straightforward to compute the commutator between $X_{S,f}$ using the fact that the left invariant vector fields X^i on $SU(2)$ satisfy $[X^i, X^j] = \epsilon^{ijk} X^k$:

$$[X_{S,f}, X_{S',f'}] \cdot C_\gamma = \frac{1}{4} \sum_{\bar{p}} f^i(\bar{p}) f'^j(\bar{p}) \epsilon_{ijk} \left(\sum_{I_{\bar{p}}^{uu'}} X_{I_{\bar{p}}^{uu'}}^k - \sum_{I_{\bar{p}}^{ud'}} X_{I_{\bar{p}}^{ud'}}^k - \sum_{I_{\bar{p}}^{du'}} X_{I_{\bar{p}}^{du'}}^k + \sum_{I_{\bar{p}}^{dd'}} X_{I_{\bar{p}}^{dd'}}^k \right) \cdot c \tag{3.7}$$

where the sum extends *only* on vertices $\bar{p}$ that lie on the *intersection* of the surfaces S and S', and, $I_{\bar{p}}^{ud'}$, for example, denotes an edge passing though point $\bar{p}$ that lie 'above' S and below S'. Thus, if γ has edges that intersect S and S' on the one-dimensional curve on which S and S' themselves intersect, the commutator will fail to vanish in general. Such intersections are of course non-generic. Nonetheless, the subspace of Cyl on which is the commutator has a non-trivial action is infinite dimensional. This non-commutativity is simply mirrored to the triad operators in quantum theory; thus the structure found in [1] is not surprising from the classical perspective.

Nonetheless, since the non-commutativity between quantum Riemannian structures is a striking feature, let us probe equation (3.7) further.

Recall, first that the functions ${}^2E[S, f]$ were obtained by a limiting procedure from the three-dimensionally smeared functions ${}^3E[f]$. Since these ${}^3E[f]$ are linear in the triad, they are associated, in a one-to-one manner with the vector fields ${}^3X[f]$ on $\mathcal{A}$:

$$ {}^3X[f] \cdot g := \int_\Sigma \mathrm{d}^3x \, f_a^i(\vec{x}) \frac{\delta g}{\delta A_a^i(\vec{x})}. \tag{3.8}$$

Although, as we saw in section II B, the action of these vector fields does not preserve Cyl, they do have a well-defined action on Cyl (and in fact preserves the space of all smooth functions on $\mathcal{A}$). Furthermore, it is clear from equation (2.7) that our fundamental vector fields $X_{f,S}$ arise as limits of ${}^3X[f]$,

$$X_{S,f} \cdot C_\gamma = \lim_{\epsilon \mapsto 0} {}^3X[{}^\epsilon f] \cdot C_\gamma, \tag{3.9}$$

for all cylindrical functions C_γ. Now, it is obvious from their definition that the vector fields ${}^3X[f]$ commute. So, it is at first puzzling that the $X_{S,f}$ do not. How does this arise? After all, we have

$$\lim_{\epsilon' \mapsto 0} \lim_{\epsilon \mapsto 0} [{}^3X[{}^{\epsilon'}f'], \, {}^3X[{}^\epsilon f]] \cdot C_\gamma = \lim_{\epsilon \mapsto 0} \lim_{\epsilon' \mapsto 0} [{}^3X[{}^{\epsilon'}f'], \, {}^3X[{}^\epsilon f]] \cdot C_\gamma = 0 \tag{3.10}$$

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for all C_γ since the commutators vanish before taking the limit. Note however, that the commutator $[X_{S',f'}, X_{S,f}]$ does not result from either of these limits. Rather, it is given by

$$[X_{S',f'}, X_{S,f}] \cdot C_\gamma$$
$$= \lim_{\epsilon' \mapsto 0} {}^3X[{}^{\epsilon'}f'] \left(\lim_{\epsilon \mapsto 0} {}^3X[{}^\epsilon f] \cdot C_\gamma \right) - \lim_{\epsilon \mapsto 0} {}^3X[{}^\epsilon f] \left(\lim_{\epsilon' \mapsto 0} {}^3X[{}^{\epsilon'}f'] \cdot C_\gamma \right) \qquad (3.11)$$

By expanding out the two terms one can see explicitly that the non-commutativity arises because, while acting on cylindrical functions, the action of the vector fields ${}^3X$ does not commute with the operation of taking limits.

To summarize, the regularization procedure which enables us to pass to $X_{S,f}$ from ${}^3X[f]$ is quite subtle and gives rise to a striking contrast between ${}^3X[f]$ and their limits $X_{S,f}$. First, the action of ${}^3X[f]$ preserves the space of smooth functions on $\mathcal{A}$ but fails to preserve the sub-space Cyl thereof. The action of $X_{S,f}$, on the other hand, leaves Cyl invariant but is not even well-defined on more general smooth functions. Secondly, since ${}^3E[f]$ depend only on the triads and not on connections, the vector fields ${}^3X[f]$ are constant with respect to the affine structure of $\mathcal{A}$. However, this correspondence does *not* hold once we bring in singular smearing fields: even though ${}^2E[S,f]$ depend only on the triads E_i^a (i.e., have no direct dependence on connections A_a^i), the vector fields $X_{S,f}$ are *not* constant. In fact they fail to commute and this non-commutativity is then directly reflected in that of the smeared triad operators in the quantum theory.

IV. DISCUSSION

Consider classical systems in which the phase space has a natural cotangent bundle structure. To quantize such a system, one can proceed in the following steps: i) Choose a preferred set of configuration variables, i.e., functions on the configuration space $\mathcal{C}$; ii) Choose a set of preferred vector fields X on $\mathcal{C}$ which are closed under the Lie bracket and whose action leaves the space of configuration variables invariant. The space of these preferred functions and vector fields is automatically endowed with a natural Lie bracket; iii) Associate with each configuration variable in the chosen set a configuration operator and with each vector field in the chosen set a momentum operator, require that their commutators be $i\hbar$ times the Lie brackets of their classical counterparts and construct the abstract operator algebra they generate; and, iv) Find a representation of this algebra. (For a more complete description, see, e.g., [25,26].) For this procedure to lead us to a useful quantum theory, however, it is necessary that the space of functions and vector fields considered be 'sufficiently large'. For example, the configuration variables together with the momentum functions defined by the vector fields should suffice to separate the points of the phase space. Only then would one have a reasonable chance of promoting a sufficiently large class of classical observables to quantum operators (modulo the usual factor ordering ambiguities.)

For systems with a finite number of degrees of freedom, it is generally straightforward to implement these steps. For example, for configuration variables one can choose smooth functions of compact support on $\mathcal{C}$ and require that the chosen vector fields also be smooth and of compact support. For systems with an infinite number of degrees of freedom, on the other hand, the choices are not so stream-lined.

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In non-perturbative quantum gravity one is guided by the invariances of the theory. Let us begin by recalling the general setting. The configuration space can be taken to be the space $\mathcal{A}$ of all smooth $SU(2)$ connections A_a^i on a three-manifold Σ. Cylindrical functions –i.e. functions which depend on the connection only through its holonomies along edges of finite graphs– suffice to separate points of $\mathcal{A}$. Since holonomies are in effect the 'raison d'être' of connections, it is then natural to choose the space Cyl of cylindrical functions as the space of configuration variables. The action of the vector fields should leave Cyl invariant. Therefore, they have to be chosen from among derivations on the ring Cyl. The requirement that the space be 'sufficiently large' leads us to consider the derivations which arise from functions on the phase space which are linear in momenta e_{ab}^i. Together, these considerations led us to the momentum functions $^2E[S, f] := \int_S e_{ab}^i f_i \mathrm{d}S^{ab}$ obtained by smearing the triads e_{ab}^i by test field f_i on *two* dimensional surfaces and the corresponding vector fields $X_{S,f}$. Thus, if we choose for vector fields the derivations on Cyl which are obtained by taking finite linear combinations of commutators between $X_{S,f}$, we satisfy all the requirements to obtain a Lie-algebra which can serve as the point of departure for quantization.

This construction is natural in the sense that it does not involve any background structures: the connection 1-forms are integrated on one-dimensional curves, while the triad two-forms are integrated over two-dimensional surfaces. However, since we are effectively smearing fields in one or two dimensions, rather than three, our elementary variables are 'singular' in a certain sense: compared to the standard procedure followed in Minkowskian field theories, we have let the smearing fields themselves be distributional. Therefore, care is needed in calculating Poisson brackets between these configuration and momentum variables. In particular, as discussed in section III B, although the three-dimensionally smeared triads *do* Poisson commute with one another, the $^2E[S, f]$ do not. Indeed, as we saw in section II C, if we simply assume commutativity, we are led to a violation of the Jacobi identity! Thus, the lower dimensional smearings do lead to features which are at first counter intuitive. However, the procedure we followed is well-defined and internally consistent and one can proceed with quantization along the steps listed in the beginning of this section. The result is precisely the quantum theory that was developed in references [1–14].

For completeness, let us now sketch how this comes about. Having chosen the preferred class of configuration variables and vector fields, it is straightforward to construct the algebra of quantum operators. To find its representations, one can first focus on the Abelian algebra of configuration operators. One can show that, in *any* representation of this algebra, the Hilbert space is the space of square-integrable functions (for some regular measure) on a certain space, $\bar{\mathcal{A}}$, which can be thought of as a 'completion' of $\mathcal{A}$ in an appropriate sense. The configuration operators act, as expected, by multiplication. If we further require that the momentum operators act simply by derivation, mirroring the action of the vector fields $X_{S,f}$ on Cyl in the classical theory, then the requirement that the two sets of operators be self-adjoint leads us to the measure $\mu_\circ$ on $\bar{\mathcal{A}}$, referred to in the Introduction. In this quantum theory, the commutators between the momentum operators simply mirror the commutators between the vector fields $X_{S,f}$ in the classical theory. Thus, there is no anomaly.

It is nonetheless striking that the smeared triad operators do not necessarily commute. For, the triads are the fundamental fields from which all Riemannian structures are built and their non-commutativity implies that geometrical quantities such as the area operators also fail to commute. Hence, quantum Riemannian geometry is now intrinsically non-

commutative. As emphasized in the Introduction, this implies, in particular, that in this approach, (at least the naive) metric representation does not exist. What is the central assumption responsible for this surprising feature? It is that the configuration variables be cylindrical functions, or, in the gauge invariant context, traces of holonomies (i.e., Wilson loops). The assumption seems rather tame at first. Furthermore, as indicated in the Introduction, it is strongly motivated by the invariances of the theory. Yet, once it is made, a series of natural steps lead us to non-commutativity of quantum Riemannian structures. These steps do involve additional assumptions but these appear to be relatively minor, and of a rather technical nature; if desired, one could weaken or justify them. Thus, it appears that the surprising features of quantum geometry –non-commutativity, the polymer-like, one-dimensional nature of fundamental excitations and the discreteness of spectra of geometric operators– can in essence be traced back to to the assumption on the configuration variables and to the gauge and diffeomorphism invariance of the theory.

In the classical and quantum theory discussed in this series of papers we have focussed on the kinematical structures. Consider, for example, the area of a fixed surface S. In the classical theory, it is a function on the full –rather than the reduced– phase space and, in the quantum theory, an operator on the kinematic –rather than the physical– Hilbert space. It does not commute with the constraints and is thus not a 'Dirac observable'. Therefore, the physical implications of the non-commutativity are not immediately transparent. To clarify this issue, let us re-examine the situation in classical general relativity. There, differential geometry provides us with a mathematical formula to compute the area of *any* surface. To relate it to physical measurements, we specify the surface operationally, typically using matter fields. It is natural to expect that the situation would be similar in the quantum theory. Given *any* surface S, quantum Riemannian geometry provides us with an operator $\hat{A}_S$. To relate it to observable quantities, we would only have to specify the surface operationally. Therefore, the result on non-commutativity of areas *should* have observable consequences: the Heisenberg uncertainty principle prevents us from measuring areas of intersecting surfaces with arbitrary accuracy.

Can one reconcile this with the fact that we have encountered no such limit in the laboratory? The answer is in the affirmative. Furthermore, the argument can be made at a sufficiently general level, without recourse to a detailed measurement theory[4], which is fortunate since such a theory is yet to be developed in quantum gravity. Consider two *macroscopic* surfaces S_i, i.e., two surfaces and a semi-classical state in which their areas are very large compared to the Planck area. Then the expectation value of their areas may be roughly estimated as $\langle A(S_i) \rangle \approx l_P^2 N(S_i) m_i$, where m_i is an 'average' spin, $N(S_i)$ is the 'effective' number of transverse intersections between the graph underlying the semi-classical state and the surface S_i, and $l_P^2 := G\hbar$ is the Planck area. In the same way, the expectation

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[4]Incidentally, such a theory will have interesting twists of its own in the gravitational context since we can no longer use 'infinitely heavy' instruments that Bohr and Rosenfeld were forced to introduce in their analysis of the Heisenberg uncertainties associated with the quantum electromagnetic field. So, a priori it is not clear that even the area of a single surface can be measured with an arbitrary accuracy. (See, e.g. [27].) For our argument, however, this subtlety is not relevant.

value of the commutator of the area operators is approximately $l_P^4 \, N(S_1 \cap S_2)k$, where k is the corresponding 'effective' spin. (In a semi-classical state, m_i and k are of the same order.) We can estimate the relative uncertainty as follows:

$$\frac{\Delta(A(S_1))}{\langle A(S_1)\rangle}\frac{\Delta(A(S_2))}{\langle A(S_2)\rangle} \geq \frac{k\,N(S_1 \cap S_2)}{m_1 m_2 \, N(S_1)N(S_2)} \tag{4.1}$$

Now, the intersection of two surfaces is a subset of measure zero in either one of the surfaces. Therefore, for a semi-classical state, $N(S_1 \cap S_2)$ must be negligible compared to $N(S_1)$ or $N(S_2)$, and the inequality should be close to being saturated, whence the relative uncertainty should also be negligible. Thus, because the commutator of two area operators is a distribution with only one-dimensional support, the uncertainties associated with their non-commutativity are completely negligible on semi-classical states.

Finally, note that the non-commutativity between geometric structures we have encountered here is quite different from that of non-commutative geometry of Connes and others. In our approach, the manifold itself is in tact, the notions of curves α, surfaces S and regions R of the three-manifold are all well-defined. (If we consider matter fields, we can even specify these objects operationally.) Hence they serve as well-defined labels for the length, area and volume operators, $\hat{\ell}_\alpha, \hat{A}_S, \hat{V}_R$, respectively. Non-commutativity refers to these operators; it refers to quantum Riemannian structures, where the term 'quantum' is used in the old-fashioned, text-book sense. In Connes' framework, by contrast, the non-commutativity occurs at a mathematically deeper level: 'the manifold itself becomes non-commutative'. More precisely, one begins with the observation that for standard manifolds, the manifold structure is coded in the *Abelian* $C^\star$-algebra of smooth functions and generalizes geometry by considering instead *non-Abelian* $C^\star$ algebras (equipped with certain additional structures). Note, however, that in this general framework, there is no 'quantization' in the traditional sense, no obvious place for $\hbar$. Once the algebra is non-commutative, 'points of the manifold disappear' and there is no obvious meaning to curves α, surfaces S or regions R and hence to the associated observables, $\hat{\ell}_\alpha, \hat{A}_S$ and $\hat{V}_R$. Thus, although in both approaches algebraic methods are used heavily to introduce geometric notions, the two meanings of 'non-commutativity' are quite different from one another.

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